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String propagation in the presence of cosmological singularities

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Abstract: We study string propagation in a spacetime with positive cosmological constant, which includes a circle whose radius approaches a finite value as $|t| \to \infty$, and goes to zero at $t = 0$. Near this cosmological singularity, the spacetime looks like $\mathbb{R}^{1,1}/\mathbb{Z}$. In string theory, this spacetime must be extended by including four additional regions, two of which are compact. The other two introduce new asymptotic regions, corresponding to early and late times, respectively. States of quantum fields in this spacetime are defined in the tensor product of the two Hilbert spaces corresponding to the early time asymptotic regions, and the S-matrix describes the evolution of such states to states in the tensor product of the two late time asymptotic regions. We show that string theory provides a unique continuation of wavefunctions past the cosmological singularities, and allows one to compute the S-matrix. The incoming vacuum evolves into an outgoing state with particles. We also discuss instabilities of asymptotically timelike linear dilaton spacetimes, and the question of holography in such spaces. Finally, we briefly comment on the relation of our results to recent discussions of de Sitter space.

Keywords: Superstring Vacua, Conformal Field Models in String Theory, Cosmology of Theories beyond the SM.

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1. Introduction

The purpose of this paper is to study some time-dependent solutions in string theory with a positive cosmological constant. At early and late times ($|t| \to \infty$), the spacetimes we will consider asymptote to linear dilaton solutions, with string frame metric and dilaton

$$\begin{aligned} ds^2 &= -dt^2 + dx^i dx^i \, ; \\ e^{\Phi} &= e^{-Q|t|} \, , \end{aligned} \tag{1.1}$$

where x^i, $i = 1, \ldots, d$ are spatial coordinates,[1] and the linear dilaton slope Q is related to the dimension of space d in a way described below. Thus, the theory becomes arbitrarily weakly coupled in the far past and far future.

The finite time behavior is non-trivial, and generically these solutions contain cosmological singularities. One of the issues of interest is the physics associated with these

[1] One can also replace the flat space labeled by x_i by a more general background, corresponding to a CFT $\mathcal{M}$ with central charge d, which is not necessarily integer.

singularities. In particular, it is in general not clear how to define observables in a cosmological spacetime: should one specify initial conditions near a singularity [1], or include a "pre-big bang" period [2] and specify initial conditions in an "asymptotic past trivial" regime? We will see below that string theory in the backgrounds (1.1) seems to favor the latter.

In fact, in the example studied in this paper, we will find that the global spacetime contains two incoming and two outgoing regions, connected to each other in a non-trivial way. String theory provides a prescription for continuing wavefunctions through the singularities separating the various regions. This allows one to study an S-matrix for string propagation through the singularities, and thereby obtain information about physics in the vicinity of the singularity. In particular, we will determine the natural incoming and outgoing vacua, and compute the Bogolubov coefficients relating them. We will find that the incoming vacuum evolves to a state with particles.

Another interesting issue concerns the stability of the solutions (1.1). In spacetimes with positive cosmological constant, SUSY is usually broken and one must check for the existence of growing modes. We will show that asymptotically timelike linear dilaton spacetimes in general contain non-negative mass squared modes whose wavefunctions grow exponentially with time at early and late times. The contribution of these modes to one loop (torus) amplitudes is infrared divergent, like that of a tachyon in flat spacetime with constant dilaton. In particular, the graviton is tachyonic in this case.

One of the motivations for this work is the question of holography in gravity with positive cosmological constant. In [3] it was proposed that gravity in de Sitter spacetime is dual to a euclidean CFT living on the boundary at $|t| \to \infty$. This duality is modeled after the AdS/CFT correspondence for negative cosmological constant. In particular, on-shell wavefunctions in de Sitter space with specified behavior near the boundary are expected to be dual to off-shell operators in the CFT, and S-matrix elements in de Sitter space should correspond to Green functions in the boundary theory. Many issues concerning this duality remain unclear. For example, one finds that while low mass fields in the bulk are dual to operators with real scaling dimensions in the boundary CFT, above a certain critical value of the mass the scaling dimensions become complex. Also, it is not clear whether the duality requires the presence of both past and future boundaries, or whether one of them is sufficient (this is related to the question of observables mentioned above).

Some of these issues can be studied in asymptotically timelike linear dilaton spacetimes, which share many properties with (global) de Sitter space. Both contain spacelike boundaries[2] at early and late times, and some properties of solutions of the wave equation near the boundary are very similar. As in the case of de Sitter space, in asymptotically timelike linear dilaton vacua there is a positive value of the mass squared at which the qualitative behavior of the solutions changes. The main advantage of timelike linear dilaton solutions is that they are easy to embed in string theory, which allows these issues to be studied in a relatively controlled setting.

[2]In the linear dilaton case, the boundary is the weak coupling region, $|t| \to \infty$, as in [4] in the spacelike case.

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Another reason to expect that timelike linear dilaton solutions should be useful for studying holography in de Sitter space is the analogy to the case of negative cosmological constant. In that case, it is known that AdS and spacelike linear dilaton vacua have a holographic description in terms of CFT [5] and Little String Theory [4], respectively. Many of the issues related to holography in AdS and spacelike linear dilaton spacetimes are similar. For example, in both cases non-normalizable on-shell bulk wavefunctions correspond to operators in the boundary theory; the Breitenlohner-Freedman bound of AdS space [6] has a precise analogue in spacelike linear dilaton theories; issues related to stability are similar in the two cases. Thus, it is natural to expect that if the AdS/CFT correspondence has an analogue in de Sitter space, something similar should happen for timelike linear dilaton spacetimes. Of course, one does not expect timelike linear dilaton spacetimes to be dual to CFT's, just as in the case of negative cosmological constant, but the more qualitative issues mentioned above should have counterparts in the linear dilaton case.

With this in mind, we discuss below the implications of our results for holography in de Sitter and timelike linear dilaton spacetimes. We find that the physical picture is rather different for positive and negative cosmological constant. The observables[3] in de Sitter space and timelike linear dilaton backgrounds are more similar to the scattering states of flat space string theory than to the non-normalizable observables familiar from AdS. Correlation functions of these observables compute S-matrix elements of the scattering states. Wavefunctions which give rise to real scaling dimensions in the boundary CFT correspond to growing modes and lead to infrared divergences in loop amplitudes.

The plan of the paper is as follows. In section 2 we discuss some classical solutions in gravity with positive cosmological constant. At large $|t|$ these solutions asymptote to (1.1). They exhibit a cosmological singularity at $t = 0$, near which they look like generalized Kasner solutions. We focus on a particular case, the generalized Milne universe, in which the geometry includes a circle whose radius shrinks from a finite value at $t = -\infty$ to zero at $t = 0$, and then increases again to the same finite value at $t = +\infty$. Near the big crunch/big bang singularity at $t = 0$ the geometry looks like $\mathbb{R}^{1,1}/\mathbb{Z}$ (and thus has regions with closed timelike curves). We point out that it is natural to continue the spacetime to one which has four asymptotic regions, two corresponding to early times, and two to late times (see figure 2).

In section 3 we embed the generalized Milne universe into string theory. We discuss the structure of the Hilbert space of asymptotic states in each of the two incoming (or, alternatively, outgoing) regions in figure 2, and show that the full spacetime can be viewed as a certain coset of $\mathrm{SL}(2,\mathbb{R})$.

In section 4 we study perturbations of the generalized Milne universe, associated with a minimally coupled scalar field of mass m, using the $\mathrm{SL}(2,\mathbb{R})$ description to continue wavefunctions through the singularities. We show that the Hilbert space of in-states is the direct product of the Hilbert spaces corresponding to the two in-regions in figure 2, and similarly for the Hilbert space of out-states. We compute the Bogolubov coefficients relating the natural creation and annihilation operators at early times to those at late times. The in-vacuum corresponds to a state with particles in the out-Hilbert space.

[3] or, in the de Sitter case, meta-observables [7].

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In section 5 we discuss the growing modes which are typically present in asymptotically timelike linear dilaton spacetimes (1.1). We show that these modes, which have $m^2 < Q^2$, give rise to infrared divergences in one loop amplitudes.

In section 6 we comment on the relation of our results to gravity in asymptotically de Sitter space, which shares many of the properties discussed in the asymptotically timelike linear dilaton case. In particular, we point out that fields with mass smaller than the Hubble mass have properties similar to those of fields with $m^2 < Q^2$ in the timelike linear dilaton case.

In section 7 we summarize our results and comment on them.

2. Some classical solutions with positive cosmological constant

We start with a discussion of some solutions of gravity coupled to a dilaton, with a positive cosmological constant. In the applications discussed in the following sections, the gravity approximation is in general invalid, because the gradient of the dilaton is typically of order one in string units, and due to the presence of (timelike and spacelike) singularities. Nevertheless, the gravity analysis provides a useful qualitative guide to the structure of spacetime. In some cases, the gravity solution does not receive stringy corrections and is exact, even in the presence of singularities. The behavior of fields near singularities is in general ambiguous in general relativity, and will be dealt with in the next sections, using ideas from string theory.

In $d+1$ dimensional spacetime, string theory gives rise to a classical low-energy effective action of the form

$$S_g = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} e^{-2\Phi} [R + 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2\Lambda] \,, \tag{2.1}$$

where $\kappa^2 = 8\pi G_N$. The fields Φ and g are the $d+1$ dimensional dilaton and metric, respectively. The other massless and massive modes of the string will be set to zero for now; we will discuss them in the next sections, when we study perturbations.[4] A non-zero tree level cosmological constant Λ can be obtained, for example, by considering non-critical strings (see e.g. [9, p. 114]), but for now we will treat it as a free parameter.

We look for solutions for which the dilaton Φ and the diagonal components of the string frame metric g depend on time but not on space:[5]

$$ds^2 = -dt^2 + \sum_{i=1}^{d} e^{2\alpha_i(t)} dx_i^2 \,. \tag{2.2}$$

If x_i is compact, it is convenient to take it to live on a circle of radius $l_s = \sqrt{\alpha'}$. $R_i(t) = l_s \exp(\alpha_i(t))$ is then the dynamical radius of the i'th dimension. In the rest of this paper we will set $l_s = 1$.

[4] Ramond-Ramond fields have recently been included in [8], with the aim of constructing de Sitter solutions in supercritical string theory.

[5] More general solutions can be obtained by applying $\mathrm{O}(d,d)$ transformations [10]–[14].

The equations of motion of $\alpha_i(t)$, $\Phi(t)$ which follow from the action (2.1) can be derived from the mini-superspace action

$$S_g = -\frac{1}{2\kappa^2}\int dt\sqrt{-g_{00}}e^{-\phi}\left(\sum_{i=1}^{d} g^{00}\dot{\alpha}_i^2 - g^{00}\dot{\phi}^2 + 2\Lambda\right), \tag{2.3}$$

where ϕ is defined by

$$\phi = 2\Phi - \sum_{i=1}^{d}\alpha_i\,. \tag{2.4}$$

One can think of $\exp(\phi/2)$ as the effective string coupling of the lower-dimensional theory obtained by averaging over x_i. The equations of motion which follow from the action (2.3) are [15, 16, 17]

$$\begin{aligned} \ddot{\alpha}_i - \dot{\alpha}_i\dot{\phi} &= 0\,;\\ 2\ddot{\phi} - \dot{\phi}^2 - \sum_{i=1}^{d}\dot{\alpha}_i{}^2 + 2\Lambda &= 0\,;\\ \sum_{i=1}^{d}\dot{\alpha}_i{}^2 - \dot{\phi}^2 + 2\Lambda &= 0\,. \end{aligned} \tag{2.5}$$

The last equation is obtained by varying with respect to g_{00}, before fixing the gauge $g_{00} = -1$. Equations (2.5) can be reduced to:

$$\begin{aligned} \dot{\alpha}_i &= c_i e^{\phi}\,;\\ \ddot{\phi} &= \dot{\phi}^2 - 2\Lambda = \sum_{i=1}^{d}\dot{\alpha}_i^2\,, \end{aligned} \tag{2.6}$$

where c_i are arbitrary (real) integration constants.

To understand the qualitative structure of the solutions of (2.6), it is useful to note that one can think of the dilaton ϕ as the position (on an infinite line) of a non-relativistic particle of unit mass. Substituting the first line of (2.6) into the second then gives the equation of motion of the particle in the potential

$$V(\phi) = -\frac{1}{2}e^{2\phi}\sum_{i=1}^{d}c_i^2\,, \tag{2.7}$$

with Λ playing the role of the total energy of the particle. We next discuss some solutions of these equations that are going to be of interest below.

2.1 Timelike linear dilaton

A simple solution of (2.6) is obtained by setting all the integration constants c_i to zero. In this case, the potential (2.7) vanishes, and the auxiliary particle moves with constant velocity determined by its total energy Λ. All α_i are constant, and the string-frame metric is flat. The "shifted" dilaton ϕ is simply twice the true dilaton (see (2.4)), and (2.6) can be solved to give

$$ds^2 = -dt^2 + dx_i^2\,, \qquad \Phi = \pm\sqrt{\frac{\Lambda}{2}}\,t\,. \tag{2.8}$$

As we review in section 3, the solution (2.8) corresponds to a simple exact CFT (for any Λ), but unfortunately it is singular. Indeed, this solution has the property that the string coupling $g_s = \exp(\Phi)$ diverges either at large positive time (for the $+$ sign in (2.8)) or at large negative time (for the $-$ sign). Thus, the physics of this model is not perturbative – correlation functions are typically pushed into the strong coupling region (see e.g. [18, 19] for discussions).

For $t \neq 0$, one can consider the solution

$$\alpha_i = \text{const}\,, \qquad \Phi = -\sqrt{\frac{\Lambda}{2}}\,|t|\,, \tag{2.9}$$

for which the string coupling does not necessarily become large anywhere. However, (2.9) does not solve the sourceless equations of motion at $t = 0$, and is thus incomplete. In subsection 2.2, we will construct solutions which look like (2.9) for large $|t|$ but resolve the apparent non-analyticity near $t = 0$.

If the dimension of spacetime is larger than two ($d > 1$), one can pass to the Einstein frame, which is useful for some purposes. Performing the standard Weyl transformation [9] on the string frame metric (2.8),

$$g_{\mu\nu,E} = e^{-\frac{4\Phi}{d-1}} g_{\mu\nu}\,, \tag{2.10}$$

one finds the line element

$$ds_E^2 = e^{\frac{4Qt}{d-1}}\left(-dt^2 + dx_i^2\right), \tag{2.11}$$

where $Q = \mp\sqrt{\Lambda/2}$. We will take Q to be positive,

$$Q = \sqrt{\frac{\Lambda}{2}}\,, \tag{2.12}$$

so that the string coupling

$$g_s = e^{-Qt} \tag{2.13}$$

goes to zero at late times. Defining a new time coordinate

$$\tau = \frac{d-1}{2Q}\,e^{\frac{2Qt}{d-1}}\,, \tag{2.14}$$

(2.11) becomes

$$ds_E^2 = -d\tau^2 + \left(\frac{2Q}{d-1}\right)^2 \tau^2 dx_i^2\,, \tag{2.15}$$

an FRW metric with a linearly growing scale factor [20]

$$a(\tau) = \frac{2Q}{d-1}\tau\,. \tag{2.16}$$

The Ricci scalar of an FRW metric with zero spatial curvature is given by

$$\mathcal{R} = 2d\frac{\ddot{a}}{a} + (d^2 - d)\left(\frac{\dot{a}}{a}\right)^2 . \tag{2.17}$$

For the metric (2.15) we find

$$\mathcal{R} = \frac{d^2 - d}{\tau^2} , \tag{2.18}$$

so that there is a curvature singularity at $\tau = 0$. Since $\tau = 0$ corresponds to $t = -\infty$ (see (2.14)), this singularity is not surprising. The string coupling (2.13) diverges as $\tau \to 0$, and one does not expect the solution in that region to be reliable.

We will mostly focus below on the string frame metric $g_{\mu\nu}$ (2.1), since this is the metric felt by fundamental strings, which will be our main focus. The Einstein frame metric is useful for comparing to solutions of general relativity, and for studying other probes in the theory.

2.2 Generalized Kasner

Since the solution with all c_i (2.6) set to zero is singular, we next turn to solutions in which some or all of the c_i are non-zero. In fact, it is not difficult to solve the equations of motion (2.6) with generic c_i. The solution has been described in [16], but in order to keep the discussion self-contained, we will review it here. We can first solve for ϕ by using the auxiliary description in terms of a particle rolling in the potential (2.7), and then substitute the solution in (2.6) to find the scale factors α_i. We furthermore impose the boundary condition that at early and late times the solution should approach the weakly coupled timelike linear dilaton one: $\alpha_i = \text{const}$, $\phi = -2Q|t|$. We find

$$\dot{\phi} = -2Q\coth(2Qt) , \tag{2.19}$$

so that

$$\phi = -\frac{1}{2}\log(\sinh^2(2Qt)) + C . \tag{2.20}$$

The effective string coupling $\exp\phi/2$ becomes large near $t = 0$, and one may expect a cosmological singularity at $t = 0$. Note also that the solution is symmetric under time reversal, $t \to -t$. Due to the singularity at $t = 0$, we are really solving the equations of motion separately for positive and negative t. The question of matching the positive and negative t solutions across the singularity will be addressed for a special case in subsections 2.3 and 3.2.

The constant of integration C in (2.20) is in principle arbitrary, but one natural way to fix it is the following. For $t \to 0$, the "velocity of the particle" $\dot{\phi}$ becomes very large and one can neglect the total energy Λ, relative to the kinetic energy $\frac{1}{2}\dot{\phi}^2$, and the potential energy $V(\phi)$, (2.7), separately. Thus, for small t one can neglect the cosmological constant Λ in the cosmological equations (2.5). The resulting equations have been studied in the context of pre-big bang scenarios (see e.g. [21, 2]); they have a well-known class of solutions corresponding to generalized Kasner or homogeneous Bianchi I spacetimes. It is natural

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to require that in the limit $t \to 0$, our solutions reduce to those of [21, 2]. This fixes C in (2.20). One finds

$$\begin{aligned} \phi &= -\frac{1}{2}\log\left[\frac{1}{4Q^2}\left(\sum_i c_i^2\right)\sinh^2(2Qt)\right]; \\ \alpha_i &= \frac{a_i}{2}\log\left[\frac{1}{Q^2}\tanh^2(Qt)\right], \end{aligned} \tag{2.21}$$

where

$$a_i \equiv \frac{c_i}{\sqrt{\sum_j c_j^2}}\,. \tag{2.22}$$

Note that a_i satisfy $\sum_i a_i^2 = 1$.

Another way of presenting the solution (2.21) is in terms of the higher dimensional dilaton Φ (2.4),

$$e^{2\Phi} = \frac{2Q}{\sqrt{\sum_j c_j^2 \sinh^2(2Qt)}}\left(\frac{1}{Q^2}\tanh^2(Qt)\right)^{\sum_i \frac{1}{2}a_i} \tag{2.23}$$

and the radii of the spatial dimensions,

$$R_i(t) = \left(\frac{1}{Q^2}\tanh^2(Qt)\right)^{\frac{1}{2}a_i}\,. \tag{2.24}$$

Depending on the sign of a_i, the i'th dimension is either expanding or contracting with time. Positive a_i corresponds to pre-big bang contraction, while negative a_i corresponds to expansion (in the string frame).

In string theory one expects

$$T_i:\ a_i \to -a_i \tag{2.25}$$

for any i to be a symmetry [22, 23]. This transformation inverts the radius (2.24) $R_i(t) \to 1/R_i(t)$, and thus acts as a time-dependent T-duality [24]. The fact that the lower dimensional dilaton ϕ (2.21) is invariant under the transformation T_i (2.25) is also consistent with this interpretation. The symmetry (2.25) is analogous to the familiar T-duality relating the euclidean cigar and trumpet [25, 26], with the radial direction of the cigar or trumpet replaced by the time, t.

2.3 Generalized Milne

For generic c_i, the solution (2.23), (2.24) has a cosmological singularity at $t = 0$. An interesting special case which is less singular is obtained by setting $c_1 = a_1 = 1$ (and thus all the other $c_i = 0$). In this case, the geometry is that of flat $d-1$ dimensional space times a two dimensional spacetime with metric [27, 16]

$$ds^2 = -dt^2 + \frac{1}{Q^2}\tanh^2(Qt)\,dx^2 \tag{2.26}$$

and dilaton

$$\Phi(t) = -\log\cosh(Qt)\,. \tag{2.27}$$

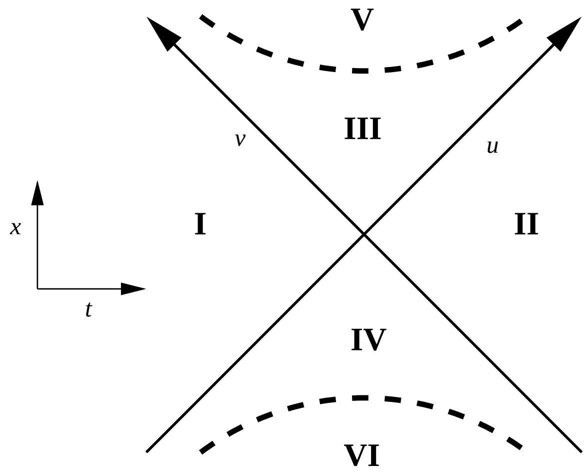

Figure 1: The generalized Milne universe.

Rescaling $t \to t/Q$, one finds

$$\begin{aligned} ds^2 &= \frac{1}{Q^2}\left(-dt^2 + \tanh^2 t\, dx^2\right); \\ \Phi &= -\log\cosh t\,. \end{aligned} \tag{2.28}$$

Near $t = 0$, (2.28) reduces to

$$\begin{aligned} ds^2 &\sim \frac{1}{Q^2}\left(-dt^2 + t^2\, dx^2\right); \\ \Phi &\sim 0\,. \end{aligned} \tag{2.29}$$

The structure of spacetime near $t = 0$ depends on whether the spatial coordinate x is compact or not.

If x is non-compact, the spacetime (2.29) is completely smooth. Indeed, the metric (2.29) is flat (see (2.17)), and the dilaton is constant. This is the two dimensional Milne universe, an unconventional parametrization of two wedges in flat Minkowski spacetime. To get the full Minkowski spacetime one must add two Rindler wedges (obtained by continuing t to imaginary values in (2.29)).

A similar continuation in the full geometry (2.28) is achieved by changing coordinates to

$$u = \sinh t\, e^{-x}\,; \qquad v = -\sinh t\, e^{x}\,, \tag{2.30}$$

in terms of which the background (2.28) is given by

$$\begin{aligned} ds^2 &= \frac{1}{Q^2}\frac{du\,dv}{1-uv}\,; \\ \Phi &= -\frac{1}{4}\log(1-uv)^2\,. \end{aligned} \tag{2.31}$$

The original (t, x) coordinates cover the two wedges $uv < 0$ (regions I, II in figure 1). The global spacetime corresponds to $-\infty < u, v < \infty$. It contains a curvature singularity and strong coupling region at $uv = 1$. Following the discussion in subsection 2.1, one might be worried that the singularity might invalidate a weak coupling treatment. We will see in section 4 that this does not seem to be the case.

The geometry (2.28), which describes regions I and II in figure 1, is invariant under translations of x; thus, it is natural to ask what happens when x is periodically identified, such that the early and late time geometry (2.28) is $\mathbb{R} \times S^1$. The resulting spacetime is singular; the spatial circle in region I shrinks from a finite size at large negative t to zero size at $t = 0$, and then expands again for positive t (in region II). Thus, the identification $x \simeq x + 2\pi R$ which is spacelike at large $|t|$, becomes null as $t \to 0$, and can be thought of as an identification by a boost. This gives rise to a spacetime of the kind discussed in [28]–[37]. In terms of the variables (2.30), which cover the whole spacetime of figure 1, the identification is

$$(u, v) \to (ue^{-\tau}, ve^{\tau}) \tag{2.32}$$

with $\tau = 2\pi R$. Near $uv = 0$, the manifold looks like $\mathbb{R}^{1,1}/\mathbb{Z}$. There are points with $uv = 0$ that are distinct but cannot be separated by open sets — the spacetime is non-Hausdorff. For instance, any point with $uv = 0$ is identified by (2.32) with points arbitrarily close to $u = v = 0$.

For $uv < 0$ (regions I, II in figure 1), the identification (2.32) is spacelike (as we saw before), while for $0 < uv < 1$ (regions III and IV in figure 1) it is timelike. Thus, the latter region contains closed timelike curves. For $uv > 1$ (regions V and VI), the identification becomes spacelike again. In fact, large positive uv corresponds to another pair of asymptotic early and late time regions.

The resulting spacetime is qualitatively of the form depicted in figure 2. This figure is intended to give a rough idea of the structure of asymptotic regions, but is less precise in the structure near the singularities. For example, the metric in region VI is in fact such that the radius of the circle is increasing as one moves in towards the singularity, and the string coupling grows as well. In drawing figure 2 we have implicitly performed a T-duality locally on that region, to bring it to a form like that of region I.

Due to the appearance of a cosmological singularity for compact x, many aspects of QFT in the spacetime (2.28), (2.31) are unclear. In particular, it is not a priori obvious whether one should restrict to the expanding, post-big bang, part of the spacetime (region II in figures 1, 2), or include the other regions indicated in figures 1, 2. In the latter case, one must understand how to continue the wavefunctions across cosmological singularities.

The approach to this problem that we will take is to realize the spacetime (2.28), (2.31) as an exact classical solution in string theory. As we will see in the next sections, string theory provides a coherent picture of the physics in the full spacetime (2.31). It describes both the "pre-big bang" regime $t < 0$ (region I) and the "post-big bang" regime $t > 0$ (region II), as well as the other regions in figures 1, 2, and gives rise to well defined matching conditions of wavefunctions across the singularities. It can thus be used to evolve fields through the singularities, and study string interactions in the spacetime (2.28), (2.31).

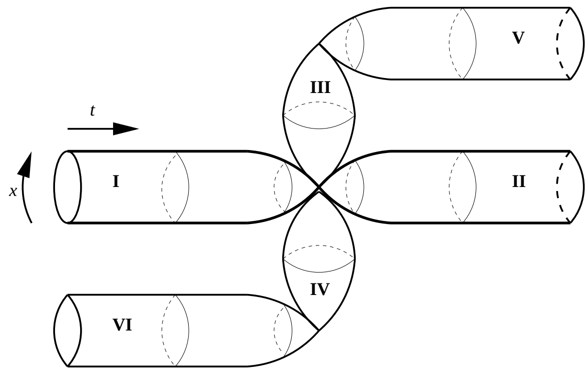

Figure 2: The periodically identified generalized Milne solution.

3. String theory

In this section we describe some features of string propagation in the spacetime (2.28), (2.31). A nice property of this spacetime is that the string coupling

$$g_s = e^{\Phi} = \frac{1}{\cosh t} \tag{3.1}$$

goes to zero at early ($t \to -\infty$) and late ($t \to +\infty$) times. This allows one to set up the incoming and outgoing Hilbert spaces in regions where interactions can be neglected; in these regions the background is described by the timelike linear dilaton solution discussed in subsection 2.1. In order to study interactions, one has to extend the solution to all t.

In subsection 3.1 we describe the $|t| \to \infty$ part of the background. In subsection 3.2 we describe the full solution, which turns out to be a coset CFT. Some properties of excitations in the spacetime (2.28) are discussed in sections 4, 5.

3.1 Timelike linear dilaton

In this subsection we review some properties of the timelike linear dilaton solution, an exact CFT describing string propagation in flat spacetime with a varying string coupling, given by (2.13). This solution is discussed e.g. in [38, 20, 39, 9].

We start with the bosonic string. The worldsheet fields $x^\mu = (t, x^i)$ are free, but the time dependence of the dilaton is reflected in a modified worldsheet stress tensor,

$$T(z) = -\partial x_i \partial x^i + \partial t \partial t - Q\partial^2 t\,. \tag{3.2}$$

The central charge is modified accordingly,

$$c = d + 1 - 6Q^2\,. \tag{3.3}$$

Since the total central charge must be equal to 26 in the bosonic string, one finds that d and Q are related,

$$6Q^2 = d - 25\,. \tag{3.4}$$

We see that asymptotically timelike linear dilaton solutions are typically *supercritical.* One can replace the d free fields x^i by an arbitrary unitary CFT, and get more general timelike linear dilaton solutions in which space is curved. For concreteness, we will mostly discuss here the case of flat space.

Since the worldsheet theory is free and the string coupling is small, it is not difficult to construct the physical states. As in flat spacetime with constant dilaton, states are labeled by momentum p_μ and an oscillator contribution. The corresponding vertex operators have the form

$$V = e^{ip_\mu x^\mu} P_N(\partial x^\mu, \bar\partial x^\mu, \ldots)\,, \tag{3.5}$$

where P_N is a polynomial in derivatives of the worldsheet fields x^μ of total (left and right) scaling dimension N, and we have taken the zero mode part of the vertex operator to be a plane wave. Physical states correspond to Virasoro primaries of the form (3.5) with scaling dimension one. From (3.2), it follows that the scaling dimension of V (3.5) is

$$L_0 = \frac{1}{4}p_i^2 - \frac{1}{4}p_0(p_0 - 2iQ) + N\,. \tag{3.6}$$

The non-standard contribution of p_0 to the scaling dimension is easy to understand. In string theory, the (zero mode parts of) vertex operators for emission of string modes have in general the form

$$V = g_s \Psi\,, \tag{3.7}$$

where Ψ is the wavefunction of the state, and g_s the string coupling. Usually, the factor of g_s in (3.7) can be neglected since it is constant, but here it is time-dependent (2.13) and therefore needs to be retained. The vertex operator (3.5) corresponds to the wavefunction

$$\Psi(\vec{x}, t) = e^{i\vec{p}\cdot\vec{x}} e^{i(p_0 - iQ)t}\,. \tag{3.8}$$

Thus, the energy is

$$E = p_0 - iQ \tag{3.9}$$

and (3.6) reads

$$L_0 = \frac{1}{4}\left(\vec{p}^2 - E^2 - Q^2\right) + N\,. \tag{3.10}$$

The mass shell condition is

$$m_{\text{eff}}^2 = E^2 - \vec{p}^2 = 4(N-1) - Q^2\,. \tag{3.11}$$

The effect of the linear dilaton is a downward shift of all masses squared by Q^2. Denoting by m^2 the combination

$$m^2 = 4(N-1)\,, \tag{3.12}$$

as in the constant dilaton case, we have

$$m_{\text{eff}}^2 = m^2 - Q^2\,. \tag{3.13}$$

For example, the graviton behaves as a particle of mass $m_{\text{eff}}^2 = -Q^2$ in the linear dilaton background.

The negative mass shift (3.11) leads to the fact that there is a finite range of positive masses squared, $0 < m^2 < Q^2$, which are effectively tachyonic. We will return to this issue in section 5 For now, we compute the torus partition sum,

$$Z(\tau) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}}) \,, \tag{3.14}$$

which summarizes the multiplicities of states at different mass levels. Here $q = e^{2\pi i \tau}$ and $\bar{q} = e^{-2\pi i \bar{\tau}}$. On a euclidean worldsheet, τ is complex, and $\bar{\tau} = \tau^*$. On a Minkowski torus, τ and $\bar{\tau}$ are independent real numbers.

It is convenient to perform the calculation in the covariant formalism. One finds

$$Z^{\mathrm{bos}}_{d+1}(\tau) = V_{d+1} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} (q\bar{q})^{-\frac{d+1-6Q^2}{24} + \frac{1}{4}(\vec{p}^2 - p_0^2 + 2iQp_0)} \left(\sum_{\mathrm{osc}} q^N \bar{q}^{\tilde{N}} \right) Z_{\mathrm{ghost}} \,. \tag{3.15}$$

Here, V_{d+1} is the volume of $d+1$ dimensional spacetime. Completing the square in the exponent and rewriting the resulting amplitude in terms of E (3.9), we find

$$Z^{\mathrm{bos}}_{d+1}(\tau) = V_{d+1} \int \frac{d^d p}{(2\pi)^d} \int \frac{dE}{2\pi} (q\bar{q})^{-\frac{d+1}{24} + \frac{1}{4}(\vec{p}^2 - E^2)} \left(\sum_{\mathrm{osc}} q^N \bar{q}^{\tilde{N}} \right) Z_{\mathrm{ghost}} \,. \tag{3.16}$$

The string amplitude involves also an integral over τ, $\bar{\tau}$. Since spacetime has lorentzian signature, one must take the worldsheet to be lorentzian as well.[6] The τ integral is then oscillatory, and one needs to supply an $i\epsilon$ prescription to compute it. It is possible to deform the contours of integration over τ and E in (3.15) such that τ becomes the modulus of a euclidean torus, and E is integrated over the imaginary axis. This is a standard procedure which is described e.g. in [9], page 83. The final answer one obtains is

$$Z^{\mathrm{bos}}_{d+1} = \int \frac{d^2\tau}{4\tau_2} Z^{\mathrm{bos}}_{d+1}(\tau) = iV_{d+1} \int \frac{d^2\tau}{16\pi^2\tau_2^2} \left(4\pi^2\tau_2\right)^{-\frac{(d-1)}{2}} (\eta\bar{\eta})^{1-d} \,. \tag{3.17}$$

Note that the partition sum (3.17) is proportional to the volume of time. This is natural from the spacetime point of view, since Z_{d+1} is a trace over free string states; it receives contributions from the infinite time period during which the string coupling is very small, and the physics is effectively time translation invariant. From the worldsheet perspective, the factor of the volume of time arises since the linear dilaton term in the worldsheet lagrangian, which breaks time translation invariance, vanishes on the torus (it is proportional to $\hat{R}$, the curvature of the worldsheet, which vanishes in this case). Thus, this amplitude is invariant under time translations.

The negative mass shift (3.11) is manifest in the partition sum (3.17). From the form of the Dedekind η function one finds that level matched states with oscillator number N contribute terms that go like

$$(q\bar{q})^{-\frac{d-1}{24} + N} = (q\bar{q})^{\frac{1}{4} m^2_{\mathrm{eff}}} \,, \tag{3.18}$$

where we used (3.4), (3.11). In particular, states with negative m^2_{eff} contribute negative powers of $q\bar{q}$, a fact that will be important in section 5, when we discuss infrared divergences.

[6] For recent discussions of lorentzian and euclidean worldsheet and spacetime, see [40, 35].

It is also worth pointing out that the negative mass shift $-Q^2$ in (3.11) is related to another interesting physical property of the model, its high energy density of states. It was shown in [41] (see also [19]) that in string theory there is a close relation between infrared instabilities and the high energy density of states (a UV-IR relation). The timelike linear dilaton background discussed here is an example of this. Since the model is supercritical, i.e. the number of space dimensions is larger than twenty five (see (3.4)), the high energy density of states implicit in (3.17) is larger than that of critical bosonic string theory. Correspondingly, the infrared instability of the model is more severe than that of $25+1$ dimensional bosonic string theory due to the downward shifts of the masses of low lying states (3.11).

We next move on to type II string theory in the timelike linear dilaton background. Much of the discussion above goes through. The worldsheet theory is now superconformal. The superconformal generators are

$$\begin{aligned} T_B &= -\partial x_i \partial x^i + \partial t \partial t - Q\partial^2 t - \frac{1}{2}\psi^\mu \partial \psi_\mu \,; \\ T_F &= i\sqrt{2}\left(\psi_i \partial x^i - \psi_t \partial t + Q\partial \psi_t\right). \end{aligned} \tag{3.19}$$

The central charge is

$$\hat{c} = \frac{2}{3}c = d + 1 - 4Q^2\,. \tag{3.20}$$

Setting $\hat{c} = 10$ for criticality, we find

$$4Q^2 = d - 9\,. \tag{3.21}$$

The main new issue in the type II case concerns the chiral GSO projection. Modular invariance and spin-statistics give rise to strong constraints that need not have solutions generically. We will not attempt to classify all possible solutions, but will illustrate the resulting structures with an example from [39].

A natural choice of a GSO projection is a separate $(-)^{F_L}$ and $(-)^{F_R}$ projection on the left and right movers, as in the critical superstring. This gives rise to the partition sum[7]

$$Z^{II}_{d+1} = iV_{d+1}\int \frac{d^2\tau}{16\pi^2\tau_2^2} Z_x^{d-1} Z_\psi^+ \bar{Z}_\psi^\pm\,, \tag{3.22}$$

where

$$Z_x = \left(4\pi^2\tau_2\right)^{-\frac{1}{2}} |\eta(\tau)|^{-2} \tag{3.23}$$

and

$$Z_\psi^\pm = \frac{1}{2}\left[Z_0^0(\tau)^{\frac{d-1}{2}} - Z_1^0(\tau)^{\frac{d-1}{2}} - Z_0^1(\tau)^{\frac{d-1}{2}} \mp Z_1^1(\tau)^{\frac{d-1}{2}}\right]. \tag{3.24}$$

The choice of $\pm$ distinguishes between type IIA and type IIB string theory. A peculiar feature of (3.22)–(3.24) is that the partition sum is only modular invariant in $d+1 = 10+16m$ dimensions, $m = 1, 2, \ldots$ The resulting theory is not spacetime supersymmetric,

[7] We use the notations of [42].

which can be seen by noting that the worldsheet fermion contribution (3.24) is not zero for $d+1 \neq 10$. For example, in the twenty six dimensional non-critical superstring vacuum,

$$Z_\psi^\pm = \frac{1}{2}\left[Z_0^0(\tau)^{12} - Z_1^0(\tau)^{12} - Z_0^1(\tau)^{12} \mp Z_1^1(\tau)^{12}\right] = 24\,. \tag{3.25}$$

Thus, there are large cancellations between (spacetime) bosons and fermions in the spectrum, but they are incomplete. The full partition sum in this case is

$$Z_{d+1=26}^{II}(\tau) = (24)^2 Z_x^{24}(\tau)\,. \tag{3.26}$$

So far we discussed some properties of the spectrum of string theory in the timelike linear dilaton background. As explained in the beginning of this section, this is sufficient for studying the spectrum of asymptotic states in the full classical string theory in the background (2.31), because the background approaches a timelike linear dilaton one at $|t| \to \infty$.

To study interactions in general, one needs to understand the full background (2.31) in string theory. Nevertheless, some interactions can be studied using the linear dilaton CFT. As is clear from figure 2, one expects two kinds of interactions in the background (2.31). One kind involves processes that take place in the part of the geometry far from the big bang/big crunch region, $t \simeq 0$. Such processes, which are known as "bulk amplitudes" (see e.g. [18, 19] for a discussion) can be studied by computing Shapiro-Virasoro type amplitudes in the timelike linear dilaton background. They are proportional to the volume of the (infinite) time interval well before or after the big bang or big crunch. We will not discuss them here, since they are rather standard, and do not shed light on the physics associated with the singularity.

Of more interest is the second kind of interactions, which occur near $t = 0$, and thus are given by amplitudes that do not go like the volume of time. To study these, one needs to understand the full background (2.31), to which we turn next.

3.2 SL(2, ℝ)/ U(1)

We would like to understand the solution (2.31) in string theory. The answer to this question is in fact known. As shown in [43], the coset CFT SL(2, ℝ)/ U(1) where one gauges a non-compact spacelike U(1) in SL(2), gives rise to a sigma model in the background[8]

$$\begin{aligned} ds^2 &= -k\frac{du\,dv}{1-uv}\,; \\ \Phi &= -\frac{1}{4}\log(1-uv)^2\,. \end{aligned} \tag{3.27}$$

k is the level of the SL(2) affine Lie algebra that enters the coset construction.[9] For bosonic strings, the background (3.27) is only valid for large k. The $1/k$ (or α') corrections are known [26, 46]. For the worldsheet supersymmetric version, which is the case of interest to

[8]For early work on generalizations to other cosets and applications to cosmology, see e.g. [23, 27, 44, 45].

[9]Actually, (3.27) with $-\infty < u, v < \infty$ describes PSL(2, ℝ)/ U(1). SL(2, ℝ)/ U(1) is a double cover of this space. We will be mostly interested in the single cover, i.e. the PSL(2) case.

us here, the background (3.27) turns out to be exact [47, 46], with k being the total level of $\widehat{\mathrm{SL}(2)}$ (which receives a contribution of $k+2$ from a bosonic SL(2) WZW model and a contribution of -2 from three free fermions in the adjoint of SL(2)). The central charge of the resulting superconformal field theory is

$$\hat{c} = 2 + \frac{4}{k} \,. \tag{3.28}$$

Comparing (2.31) to (3.27), we see that the generalized Milne universe described in subsection 2.3 corresponds to a negative level SL(2)/ U(1),

$$\frac{1}{k} = -Q^2 \,. \tag{3.29}$$

Thus, the SL(2)/ U(1) CFT has central charge $\hat{c} < 2$. It is also known that in this case the $N = 1$ superconformal symmetry of the coset model is enhanced to $N = 2$. For unitary CFT's these two facts would mean that the model must be an $N = 2$ minimal model, and k must be integer. However, in our case there is no reason to impose unitarity on the CFT, since the SL(2)/ U(1) directions include time.

The background (2.31) can be thought of as a double Wick rotated version of the two dimensional black hole. This is reflected in the Penrose diagram of figure 1. The region outside the horizon of the black hole corresponds in our case to the early time epoch (region I in figure 1), while the other asymptotic region in the black hole case corresponds to the late time region II. The regions between the horizon and singularity of the black and white holes correspond in our case to regions III and IV. The singularities at $uv = 1$ appear timelike in regions III, IV. Finally, the regions behind the black and white hole singularities, V and VI, give rise to additional asymptotic regions.

In section 2 we discussed a background obtained from (2.31) by the identification (2.32). This spacetime can be obtained from $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ by further modding out by a discrete group. In terms of the $\mathrm{SL}(2,\mathbb{R})$ matrices g, the background (2.31) is obtained by gauging the continuous symmetry

$$g \to e^{\rho\sigma_3} g e^{\rho\sigma_3} \,. \tag{3.30}$$

The identified generalized Milne universe discussed in subsection 2.3 is obtained by further gauging the discrete subgroup isomorphic to $\mathbb{Z}$, which is generated by

$$g \to e^{\tau\sigma_3} g e^{-\tau\sigma_3} \,, \tag{3.31}$$

where $\tau = \pi R$ (note the factor of two relative to (2.32)).

A note on T-duality: when constructing the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ model, one has a choice of gauging either the vector symmetry (3.30), or the axial symmetry $g \to \exp(\rho\sigma_3) g \exp(-\rho\sigma_3)$. The CFT is self-dual under this replacement [25, 26, 48], but duality exchanges regions I and VI, as well as regions II and V. Regions III and IV are invariant. This will play a role in our discussion in section 4

In the next section we turn to the study of excitations propagating in the background (2.31). We will use the description of the singular spacetime of figure 2 as a coset of PSL(2), to continue wavefunctions between different regions separated by cosmological singularities, and study the resulting S-matrix.

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4. Scalar perturbations

We would like to analyze the propagation of small perturbations in the background (2.28), (2.31). For concreteness, we will consider a minimally coupled scalar field T of mass m, which could (for example) be one of the perturbative string modes described in the previous section. We will focus on the non-trivial two dimensional part of the spacetime. The other $d-1$ dimensions are flat; it is easy to incorporate them into the discussion.

The contribution of T to the action (2.1) is

$$S_T = \frac{1}{2\kappa^2}\int dxdt\sqrt{-g}e^{-2\Phi}\left(-g^{\mu\nu}\partial_\mu T\partial_\nu T - m^2T^2\right), \tag{4.1}$$

where we are neglecting self-interaction terms, as well as interactions with other fields, and focus on free propagation in the geometry (2.28), (2.31).

Since the background (2.28) is translation invariant in x, we will consider modes with a given value of spatial momentum,

$$T(t,x) = T(t)e^{ipx} + \text{c.c.} \tag{4.2}$$

Substituting this ansatz into the action (4.1), and adding (2.3), one finds

$$S_g + S_T = -\frac{1}{2\kappa^2}\int dt\sqrt{-g_{00}}e^{-\phi}\left[g^{00}(\dot\alpha^2 - \dot\phi^2 + \dot T^2) + 2\Lambda + m^2T^2 + e^{-2\alpha(t)}p^2T^2\right]. \tag{4.3}$$

The resulting equations of motion are (compare to (2.6))

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{-\phi}\dot\alpha\right) &= p^2e^{-2\alpha(t)}T^2\,;\\
\ddot\phi = \dot\phi^2 - 2\Lambda - \left(m^2 + p^2e^{-2\alpha(t)}\right)T^2 &= \dot\alpha^2 + \dot T^2\,;\\
\ddot T - \dot\phi\dot T + \left(m^2 + p^2e^{-2\alpha(t)}\right)T &= 0\,.
\end{aligned} \tag{4.4}$$

If the scalar field T is small everywhere, one can neglect the $O(T^2)$ back-reaction of the metric and dilaton, which is summarized by the first two equations in (4.4), and solve the linearized equation of motion of T given by the last line of (4.4), in the fixed background (2.26), (2.31).

Substituting the values of α and ϕ corresponding to the solution (2.26), (2.27), and defining

$$z = -\sinh^2(Qt)\,, \qquad M = \frac{m}{Q}\,, \tag{4.5}$$

brings the last line of (4.4) to the form

$$z(1-z)T'' + (1-2z)T' - \frac{1}{4}\left(M^2 - p^2\frac{1-z}{z}\right)T = 0\,, \tag{4.6}$$

where primes denote derivatives with respect to z. The asymptotic past and future regions, with $|t|\to\infty$, correspond to

$$z \simeq -\left(\frac{1}{4}\right)\exp(2Q|t|) \to -\infty\,. \tag{4.7}$$

In this limit (4.6) simplifies:

$$-z^2T'' - 2zT' - \frac{1}{4}\left(M^2 + p^2\right)T = 0\,, \tag{4.8}$$

with the solution

$$\begin{aligned} T &= (-z)^\lambda\,; \\ \lambda &= \frac{1}{2}\left(-1 \pm \sqrt{1 - M^2 - p^2}\right)\,. \end{aligned} \tag{4.9}$$

For $M^2 + p^2 > 1$, this corresponds to plane waves with energy

$$E = Q\sqrt{M^2 + p^2 - 1}\,. \tag{4.10}$$

For smaller masses one gets growing and decaying wavefunctions, as discussed in subsection 3.1. The bound on M in (4.9) is the same as the one that follows from (3.13). In this section we will concentrate on the plane waves (with $E \in \mathbb{R}$). We will return to the growing solutions in section 5

Equation (4.6) can be solved in terms of hypergeometric functions, but it is more instructive to proceed by using the construction of the space (2.31) as a coset of $\mathrm{PSL}(2,\mathbb{R})$. This provides a unified description of the behavior of the wavefunctions in all six regions in figures 1, 2.

One starts by constructing eigenfunctions of the laplacian on $\mathrm{PSL}(2,\mathbb{R})$, in which the $\mathrm{U}(1)_L\times\mathrm{U}(1)_R$ symmetries $g \to \exp(\rho_L\sigma_3)g\exp(\rho_R\sigma_3)$ are diagonal. This is described in [49] (see [36] for a review and an application in a related context), so we will be brief. The eigenfunctions of the laplacian are matrix elements of group elements between states belonging to representations of $\mathrm{PSL}(2,\mathbb{R})$. The scattering states correspond to matrix elements in the principal continuous series. States in these representations are labeled by three quantum numbers, j; m; $\pm$. The quantum number $j = -1/2 + is$, $s \in \mathbb{R}$, is related to the value of the Casimir, $j(j+1)$. The label m determines the eigenvalue under the non-compact U(1) symmetry $\exp(\alpha\sigma_3)$; for unitary representations, this eigenvalue is $\exp(2im\alpha)$, with $m \in \mathbb{R}$. The third quantum number is a discrete label which takes two values.

A generic group element $g \in \mathrm{PSL}(2,\mathbb{R})$ (with all elements of the 2×2 matrix g non-vanishing) can be written as

$$g(\alpha,\beta,\theta;\epsilon_2,\delta) = e^{\alpha\sigma_3}(i\sigma_2)^{\epsilon_2}g_\delta(\theta)e^{\beta\sigma_3}\,, \tag{4.11}$$

where $\epsilon_2 = 0, 1$; $\delta = I, II, IV$, and

$$g_{IV} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \qquad 0 \le \theta \le \frac{\pi}{2}\,, \tag{4.12}$$

$$g_I = g_{II}^{-1} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}; \qquad 0 \le \theta < \infty\,. \tag{4.13}$$

The non-vanishing matrix elements of g (4.11) in the principal continuous series are

$$K_{\pm\pm}(\lambda,\mu;j;g) \equiv \langle j,m,\pm|g|j,\bar{m},\pm\rangle = e^{2i(m\alpha+\bar{m}\beta)}\langle j,m,\pm|(i\sigma_2)^{\epsilon_2}g_\delta(\theta)|j,\bar{m},\pm\rangle\,, \tag{4.14}$$

where

$$\lambda \equiv -im - j\,; \qquad \mu \equiv -i\bar{m} - j\,; \qquad j = -\frac{1}{2} + is\,. \tag{4.15}$$

The labels m, $\bar{m}$, s take arbitrary real values. The functions (4.14) appear in [49] (see also [36]). We will present some of them later.

Given the eigenfunctions of the laplacian on $\mathrm{PSL}(2,\mathbb{R})$, we can find the wavefunctions on the coset obtained by gauging (3.30), by restricting to gauge invariant wavefunctions. Since the parameters α, β in (4.11) transform under (3.30) as $\alpha \to \alpha + \rho$, $\beta \to \beta + \rho$, the invariant wavefunctions are those with $m = -\bar{m}$. Further modding out by the identification (3.31) leads to quantization of m, $mR \in \frac{1}{2}\mathbb{Z}$. Also, twisted sectors appear, in which $m \neq -\bar{m}$. These correspond to winding modes on the coset manifold.

Thus, the matrix elements (4.14) give rise to solutions of the wave equation on the spacetime (2.31). Different regions in figures 1, 2 correspond[10] to different types of $\mathrm{PSL}(2,\mathbb{R})$ matrices g, (4.11). For example, region I in figure 1 corresponds to $\epsilon_2 = 0$, $\delta = I$ in (4.11). The coordinates on region I in (4.11), $\alpha - \beta$ and θ, are related to the coordinates x and t in (2.28) as follows:

$$\begin{aligned} x &\leftrightarrow \alpha - \beta\,; \\ t &\leftrightarrow \theta\,. \end{aligned} \tag{4.16}$$

The two independent solutions of (4.6) in region I can be taken to be [49]

$$T_I^{(+)}(t) = K_{++}(\lambda,\mu;j;g_I) = \frac{1}{2\pi i} B(\lambda, -\lambda - 2j) \frac{(1-z)^{j+\frac{\lambda+\mu}{2}}}{(-z)^{\frac{\lambda+\mu}{2}}} F(\lambda,\mu;-2j;\frac{1}{z}) \tag{4.17}$$

and

$$\begin{aligned} T_I^{(-)}(t) &= K_{--}(\lambda,\mu;j;g_I) \qquad\qquad (4.18)\\ &= \frac{1}{2\pi i} B(1-\mu, \mu+2j+1) \frac{(1-z)^{j+\frac{\lambda+\mu}{2}}}{(-z)^{2j+1+\frac{\lambda+\mu}{2}}} F\left(\lambda+2j+1, \mu+2j+1; 2j+2; \frac{1}{z}\right), \end{aligned}$$

where z is given in (4.5), $B(a,b)$ is the Euler Beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \tag{4.19}$$

and $F(a,b;c;x)$ is the hypergeometric function ${}_2F_1$. By comparing the asymptotic behavior of the two solutions (4.17), (4.18) to (4.9), one can match the parameters s, m in (4.15) with those in (4.9):

$$\begin{aligned} m &= -\bar{m} = \frac{p}{2}\,; \\ 2s &= \sqrt{M^2 + p^2 - 1} = \frac{E}{Q}\,. \end{aligned} \tag{4.20}$$

[10]This is described in more detail in [36]. The regions in figure 1 are related to those in [36, figure 2] as follows. Our regions (I, II, III, IV, V, VI) correspond to regions $(1, 1', II, I, 2', 4)$ in [36].

For the case of compact x, there are both momentum and winding modes, with the spectrum

$$\begin{aligned} m &= \frac{1}{2}\left(\frac{n}{R} + kwR\right) ; \\ \bar{m} &= \frac{1}{2}\left(-\frac{n}{R} + kwR\right) . \end{aligned} \tag{4.21}$$

Here, $n, w \in \mathbb{Z}$ are the momentum and winding, respectively, while k is the level of $\widehat{\mathrm{SL}(2)}$.

Since the wavefunctions $K_{\pm\pm}$ are given by matrix elements on $\mathrm{PSL}(2,\mathbb{R})$, (4.14), they are uniquely defined in all regions in figure 1. For example, the solution $T_I^{(+)}(t)$, which in region I is given by (4.17), can be determined in regions II, VI, by using the fact that [49, 36]

$$\begin{aligned} K_{++}(\lambda,\mu;j;g_{II}) &= K_{--}(\lambda,\mu;j;g_I) ; \\ K_{++}(\lambda,\mu;j;g_{VI}) &= 0 , \end{aligned} \tag{4.22}$$

where $g_{II} = g_I^{-1}$, as in (4.13), and $g_{VI} = -g_I i\sigma_2$. Another region that will be of interest below is region V, where

$$\begin{aligned} K_{++}(\lambda,\mu;j;g_V) = \frac{1}{2\pi i}(1-z)^{\frac{\mu-\lambda}{2}} \times \\ \times \Big[B(-2j-\lambda,1-\mu)(-z)^{-\frac{\lambda+\mu+2j}{2}} F(-\lambda-2j,1-\lambda;-\mu-\lambda-2j+1;z) + \\ + B(\lambda,\mu+2j+1)(-z)^{\frac{\lambda+\mu+2j}{2}} F(\mu,\mu+2j+1;\mu+\lambda+2j+1;z)\Big] \end{aligned} \tag{4.23}$$

with $g_V = i\sigma_2 g_I$.

Similarly, one can compute the other three eigenfunctions of the laplacian K_{+-}, K_{-+}, K_{--} in all six regions of figure 1. Of course, in any given region, only two of the four functions $K_{\pm\pm}$ are independent. The others can be expressed as linear combinations of any linearly independent two (and in some cases vanish, such as K_{++} in region VI (4.22)). We will give some additional examples below.

We are now ready to return to the problem of quantum field theory in the background (2.31). Naively, in a spacetime with metric (2.26), one would expect to specify an initial state at $t \to -\infty$ and calculate the amplitude for it to propagate to some particular final state at $t \to +\infty$. However, extending the spacetime as in (2.31) and figures 1, 2, it is clear that the initial time surface actually contains two independent components, corresponding to early times in regions I (which is covered by the original coordinates (2.26)) and VI. Similarly, the out-region is the union of late times in regions II and V. Thus, the incoming Hilbert space is expected to be the direct product of the Hilbert spaces corresponding to regions I and VI, while the outgoing Hilbert space is expected to be the direct product of Hilbert spaces corresponding to regions II and V. This picture is further supported by the fact that regions I and VI are exchanged by T-duality [25, 26], and so are regions II and V.

To quantize the field T (4.1), we would like to expand it in a complete set of orthonormal mode solutions. In the early and late time regimes $|t| \to \infty$, the metric becomes

Minkowski (1.1), and the string coupling goes to zero and so can be neglected. Therefore, it is natural to treat $i\partial/\partial t$ as the timelike Killing vector with respect to which one measures energies (as we have done in section 3). There are two natural sets of modes in terms of which the quantum field T can be expanded, corresponding to the early and late time regimes, respectively. We next construct these modes.

At early times, we are looking for two types of solutions. The first type corresponds to wavefunctions which vanish in region VI, and contain only positive frequency modes as $t \to -\infty$ in region I. Wavefunctions of the second type vanish in region I and contain only positive frequency modes at early times in region VI.

The mode solutions of the first type are $K_{++}(\lambda, \mu; j; g)$ with $s > 0$ in (4.15). This solution indeed vanishes in region VI (see (4.22)), and its $t \to -\infty$ behavior in region I is determined by (4.7), (4.17):

$$K_{++}(\lambda, \mu; j; g_I)(t \to -\infty) \simeq \frac{1}{2\pi i} B(\lambda, -\lambda - 2j)(-z)^j \simeq \frac{1}{2\pi i} B(\lambda, -\lambda - 2j) 4^{-j} e^{-2Qtj} . \tag{4.24}$$

Substituting $j = -\frac{1}{2} + is$ we have

$$K_{++}(\lambda, \mu; j; g_I)(t \to -\infty) \simeq \frac{1}{2\pi i} B(\lambda, -\lambda - 2j) 4^{-j} e^{Qt} e^{-2iQst} . \tag{4.25}$$

The factor $\exp(Qt)$ is the string coupling (see (1.1), (3.7)), while $\exp(-2iQst) = \exp(-iEt)$ (see (4.20)). Thus, $K_{++}(\lambda, \mu; j; g)$ contains only positive frequency modes at early times in region I.

Similarly, the positive energy solutions of the second type correspond to $K_{+-}(\lambda, \mu, j, g)$. One can show [49] that K_{+-} satisfies the following properties:

$$\begin{aligned} K_{+-}(\lambda, \mu, j, g_I) &= K_{++}(-\lambda - 2j, \mu, j, g_{VI}) = 0 ; \\ K_{+-}(\lambda, \mu, j, g_{VI}) &= K_{++}(-\lambda - 2j, \mu, j, g_I) , \end{aligned} \tag{4.26}$$

where we recall that $g_{VI} = -g_I i\sigma_2$. The first line of (4.26) shows that the mode $K_{+-}(t \to -\infty)$ lives purely in region VI. The second line gives the profile of K_{+-} in region VI, and in particular establishes that it corresponds to a positive energy solution as $t \to -\infty$ in this region.

One can expand the quantum field T in the above modes,

$$\begin{aligned} N\, T(\lambda, \mu) = {} & a_I(\lambda, \mu) K_{++}(\lambda, \mu; j; g) + a_I^\dagger(\lambda, \mu) K_{++}^*(\lambda, \mu; j; g) + \\ & + a_{VI}(\lambda, \mu) K_{+-}(\lambda, \mu; j; g) + a_{VI}^\dagger(\lambda, \mu) K_{+-}^*(\lambda, \mu; j; g) , \end{aligned} \tag{4.27}$$

where N is a normalization factor, which can be read off from (4.25). The vacuum $|0\rangle_{\rm in}$ is defined to be the state annihilated by the annihilation operators in regions I and VI, a_I and a_{VI}. The operators $a_I^\dagger$ and $a_{VI}^\dagger$ create particles in regions I and VI, respectively. The fact that the modes K_{++} and K_{+-}, corresponding to the two in-regions I and VI, are orthogonal in the Klein-Gordon norm (because either one or the other vanishes at all points on an early time Cauchy surface), implies that the creation and annihilation operators $a_I, a_I^\dagger$ commute with $a_{VI}, a_{VI}^\dagger$.

At late times, it is more natural to choose modes that vanish in one of the out-regions and are purely positive frequency in the other one. Using results in [49] one finds that the solution of the wave equation that vanishes in region V and has purely positive energy in region II is $K^*_{--}(-\lambda-2j,-\mu-2j;j;g)$, while the positive energy solution in region V which vanishes in region II is $K^*_{-+}(-\lambda-2j,-\mu-2j;j;g)$. Therefore, the expansion appropriate for late times is

$$\begin{aligned} NT(\lambda,\mu) = & \, a_{II}(\lambda,\mu)K^*_{--}(-\lambda-2j,-\mu-2j;j;g) + \\ & + a^\dagger_{II}(\lambda,\mu)K_{--}(-\lambda-2j,-\mu-2j;j;g) + \\ & + a_V(\lambda,\mu)K^*_{-+}(-\lambda-2j,-\mu-2j;j;g) + \\ & + a^\dagger_V(\lambda,\mu)K_{-+}(-\lambda-2j,-\mu-2j;j;g)\,. \end{aligned} \tag{4.28}$$

The out-vacuum $|0\rangle_{\rm out}$ is by definition annihilated by the annihilation operators a_{II} and a_V.

To determine the relation between the incoming and outgoing vacua, one needs to expand the outgoing modes K_{--}, K_{-+} (4.28) in terms of the incoming ones K_{++}, K_{+-} (4.27). This can be done, for example, by matching the asymptotic behaviors at early times (in regions I and VI). In region I, the asymptotic behavior of the mode solutions is

$$\begin{aligned} K_{++}(\lambda,\mu;j;g_I) & \sim \frac{1}{2\pi i}B(\lambda,-\lambda-2j)(-z)^j\,; \\ K_{+-}(\lambda,\mu;j;g_I) & = 0\,; \\ K_{--}(\lambda,\mu;j;g_I) & \sim \frac{1}{2\pi i}B(1-\mu,\mu+2j+1)(-z)^{-j-1}\,; \\ K_{-+}(\lambda,\mu;j;g_I) & \sim \frac{1}{2\pi i}\Big[B(2j+1,-2j-\lambda)+B(2j+1,\lambda)\Big](-z)^j + \\ & \quad + \frac{1}{2\pi i}\Big[B(-2j-1,\mu+2j+1)+B(-2j-1,1-\mu)\Big](-z)^{-j-1}\,. \end{aligned} \tag{4.29}$$

In region VI, it is

$$\begin{aligned} K_{+-}(\lambda,\mu;j;g_{VI}) & \sim \frac{1}{2\pi i}B(\lambda,-\lambda-2j)(-z)^j\,; \\ K_{++}(\lambda,\mu;j;g_{VI}) & = 0\,; \\ K_{-+}(\lambda,\mu;j;g_{VI}) & \sim \frac{1}{2\pi i}B(1-\mu,\mu+2j+1)(-z)^{-j-1}\,; \\ K_{--}(\lambda,\mu;j;g_{VI}) & \sim \frac{1}{2\pi i}\Big[B(2j+1,-2j-\lambda)+B(2j+1,\lambda)\Big](-z)^j + \\ & \quad + \frac{1}{2\pi i}\Big[B(-2j-1,\mu+2j+1)+B(-2j-1,1-\mu)\Big](-z)^{-j-1}\,. \end{aligned} \tag{4.30}$$

From (4.29) and (4.30), one obtains

$$\begin{pmatrix} K_{--} \\ K_{-+} \\ K^*_{--} \\ K^*_{-+} \end{pmatrix} = \begin{pmatrix} A & C & 0 & B \\ C & A & B & 0 \\ 0 & B^* & A^* & C^* \\ B^* & 0 & C^* & A^* \end{pmatrix} \begin{pmatrix} K^*_{++} \\ K^*_{+-} \\ K_{++} \\ K_{+-} \end{pmatrix}, \tag{4.31}$$

with

$$\begin{aligned} A &= -\frac{B(1-\mu,\mu+2j+1)}{B(1-\lambda,\lambda+2j+1)}\,; \\ B &= \frac{B(2j+1,-2j-\lambda)+B(2j+1,\lambda)}{B(\lambda,-\lambda-2j)}\,; \\ C &= -\frac{B(-2j-1,\mu+2j+1)+B(-2j-1,1-\mu)}{B(1-\lambda,\lambda+2j+1)}\,. \end{aligned} \tag{4.32}$$

From (4.27), (4.28) and (4.31), the following Bogolubov transformation follows:

$$\begin{pmatrix} a_I^\dagger \\ a_{VI}^\dagger \\ a_I \\ a_{VI} \end{pmatrix} = \begin{pmatrix} A & C & 0 & B^* \\ C & A & B^* & 0 \\ 0 & B & A^* & C^* \\ B & 0 & C^* & A^* \end{pmatrix} \begin{pmatrix} a_{II}^\dagger \\ a_V^\dagger \\ a_{II} \\ a_V \end{pmatrix}. \tag{4.33}$$

The Bogolubov coefficients in (4.33) satisfy the relations

$$\begin{aligned} |A|^2+|C|^2-|B|^2 &= 1\,; \\ AC^*+A^*C &= 0\,, \end{aligned} \tag{4.34}$$

which are equivalent to equations (3.39) and (3.40) in [50]. The relations (4.34) imply that (4.33) can be inverted to

$$\begin{pmatrix} a_{II}^\dagger \\ a_V^\dagger \\ a_{II} \\ a_V \end{pmatrix} = \begin{pmatrix} A^* & C^* & 0 & -B^* \\ C^* & A^* & -B^* & 0 \\ 0 & -B & A & C \\ -B & 0 & C & A \end{pmatrix} \begin{pmatrix} a_I^\dagger \\ a_{VI}^\dagger \\ a_I \\ a_{VI} \end{pmatrix}. \tag{4.35}$$

The Bogolubov coefficients can be used to determine some of the S-matrix elements of the field T, as in [50]. In particular, the fact that $B \neq 0$ signals particle creation (see [51, 37] for recent discussions of particle creation in string theory).

The incoming vacuum evolves to an outgoing state which contains

$${}_{\rm in}\langle 0|a_{II}^\dagger a_{II}|0\rangle_{\rm in} = {}_{\rm in}\langle 0|a_V^\dagger a_V|0\rangle_{\rm in} = |B|^2 \tag{4.36}$$

particles in region II as well as region V, in the mode labeled by λ, μ, s. Plugging in the explicit form of B, given in (4.32), one finds that

$$|B|^2 = \frac{\cosh^2 \pi m}{\sinh^2 \pi s}\,, \tag{4.37}$$

where we have used (4.15); m and s are related to the energy, momentum and winding via (4.20) and (4.21).

We finish this section with a few comments. Our analysis of Bogolubov coefficients leading to (4.33) is strictly speaking only valid in the large $|k|$ approximation.[11] While we have not studied the $1/k$ corrections in detail, experience with AdS_3 [52] suggests that

[11] Recall that k is the level of $\widehat{\rm SL(2)}$ entering the construction (see subsection 3.2).

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the only difference with respect to the classical analysis is the appearance of a "reflection coefficient" relating vertex operators corresponding to the representations $|j\rangle$ and $|-j-1\rangle$. One expects in general to have

$$\tilde{\theta}(-j-1)=\frac{\Gamma(1-\frac{2j+1}{k})}{\Gamma(1+\frac{2j+1}{k})}\theta(j)\,, \qquad (4.38)$$

where $\theta(j)$, $\tilde{\theta}(-j-1)$ are the quantum vertex operators corresponding to the classical wavefunctions which satisfy the $|k|\to\infty$ limit of (4.38), $\tilde{\theta}(-j-1)=\theta(j)$. This should be checked in more detail, but if true, it implies that the $1/k$ corrections modify (4.33) in the following simple way: A and C (4.32) are multiplied by $\Gamma(1-\frac{2j+1}{k})/\Gamma(1+\frac{2j+1}{k})$, while B remains the same. Since for $j=-1/2+is$, the correction factor is a phase, it does not modify the consistency conditions (4.34).

Another interesting extension of the analysis of this section involves the calculation of higher point functions in the cosmological spacetime (2.31). This can in principle be done by using results on $\mathrm{SL}(2,\mathbb{R})$ [52, 53]. For example, to calculate three point functions one needs to perform a transform of the $\mathrm{SL}(2,\mathbb{R})$ correlation functions computed in [52]. A similar transform was discussed in the context of Little String Theory in [54]. It seems to be well behaved there; more work is needed to establish whether the three point functions are also well behaved here.

Four and higher point functions are of course of interest as well, and one may hope that they can be understood using the underlying $\mathrm{SL}(2,\mathbb{R})$ affine Lie algebra structure. Of particular interest are interactions that take place near the singularities in figure 2. The amplitudes that are sensitive to such interactions should receive contributions from principal discrete series and degenerate representations of $\mathrm{SL}(2)$, which correspond to wavefunctions localized there, and can appear as intermediate states in four and higher point functions.

5. Growing modes and infrared divergences in asymptotically timelike linear dilaton spacetimes

In the previous section we studied perturbations of the generalized Milne universe (2.31) which correspond to particles with real energy E (4.10). As we noted after (4.10), for sufficiently low mass, $m^2<Q^2$, the energy is imaginary (for low enough momentum), so that the wavefunctions grow or decay exponentially with time. In this section, we will briefly discuss such modes, first classically, and then at one loop.

5.1 Classical analysis

Consider the classical evolution of a scalar field T with mass m in the linear dilaton background (1.1). For concreteness, we will first discuss the early time epoch, during which $g_s=\exp(Qt)$ (for late times, one simply replaces Q by $-Q$ in the following equations). The lagrangian (4.1) is proportional to

$$\mathcal{L}=\frac{1}{2}e^{-2Qt}\left(-\eta^{\mu\nu}\partial_\mu T\partial_\nu T-m^2T^2\right). \qquad (5.1)$$

The equation of motion of T is

$$\left(-\eta^{\mu\nu}\partial_\mu\partial_\nu + 2Q\partial_t + m^2\right) T = 0\,. \tag{5.2}$$

A basis of solutions is given by

$$T(t,\vec{x}) = e^{Qt}e^{\pm iEt + i\vec{k}\cdot\vec{x}}\,, \tag{5.3}$$

with

$$E = \sqrt{\vec{k}^2 + m^2 - Q^2}\,. \tag{5.4}$$

As discussed in section 3, it is natural to define the wavefunction Ψ to be the field T with the factor of the coupling stripped off. While this was described in the context of string theory, it is very natural in field theory as well [55]. The resulting wavefunction,

$$\Psi(t,\vec{x}) = e^{-Qt}T(t,\vec{x}) = e^{\pm iEt + i\vec{k}\cdot\vec{x}} \tag{5.5}$$

has the property that for $\vec{k}^2 + m^2 - Q^2 < 0$, it is not a plane wave, but an exponentially growing or decaying function of t,

$$\Psi(t,\vec{x}) = e^{\pm\omega t + i\vec{k}\cdot\vec{x}} \tag{5.6}$$

with $\omega = -iE$ (5.4).

One consequence of this fact is the following. Rewrite the lagrangian (5.1) in terms of the wavefunction Ψ, by using (5.5). This is natural since at early times Ψ is canonically normalized, while T has a time dependent kinetic term (5.1). Omitting a total derivative which does not influence the classical equations of motion, one finds

$$\mathcal{L} = \frac{1}{2}\left(-\eta^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - m_{\text{eff}}^2\Psi^2\right), \tag{5.7}$$

where m_{eff} is given by (3.13). This form makes it clear that the the physics at very early times is invariant under time translations, and that fields with negative m_{eff}^2 behave like tachyons in flat space with constant dilaton. For example, the hamiltonian corresponding to (5.7),

$$H = \frac{1}{2}\int d^dx\left(\dot{\Psi}^2 + (\nabla\Psi)^2 + m_{\text{eff}}^2\Psi^2\right), \tag{5.8}$$

is not constant for fields with $m_{\text{eff}}^2 + \vec{k}^2 < 0$: rather, it grows or decays exponentially with time.

Since fields with $m_{\text{eff}}^2 < 0$ look like tachyons (5.7), it is natural to ask whether they lead to instabilities of the space. Classically, this is the questions whether as one evolves the growing wavefunctions (5.6) forward in time, the self-interactions of the field T and its interactions with other fields, such as the metric and dilaton, lead to large changes in the solution (2.31) even if the initial perturbation is very small.

Consider first a spacetime that approaches an asymptotically timelike linear dilaton solution as $t \to +\infty$. The question whether such a solution is stable against perturbations by modes with $m_{\text{eff}}^2 < 0$ involves a competition between two effects: the growth of the wavefunction, $\Psi \sim \exp(\omega t)$, and the decrease of the coupling, $g_s = \exp(-Qt)$. One way to quantify this is to think about the effect of interaction terms in the lagrangian (5.1) (with $Q \to -Q$). The schematic structure of the full interacting lagrangian describing the field T is[12]

$$\mathcal{L} = \frac{1}{2} e^{2Qt} (a_2 T^2 + a_3 T^3 + a_4 T^4 + \cdots) \,. \tag{5.9}$$

In terms of the wavefunctions $\Psi = T \exp(Qt)$, (5.9) can be written as

$$\mathcal{L} = (b_2 \Psi^2 + b_3 g_s \Psi^3 + b_4 g_s^2 \Psi^4 + \cdots) \,. \tag{5.10}$$

Clearly, the effect of cubic and higher terms, relative to the leading, quadratic, term in the action is governed in the weak coupling region by the size of T. Thus, if T diverges as $t \to +\infty$, the late time solution is unstable against such perturbations, and one has to take into account tachyon condensation.

It is not difficult to see that if the mass of the scalar field T satisfies $m^2 > 0$, both solutions for T (5.3) go to zero as $t \to +\infty$, i.e. the late time behavior is stable. On the other hand, for $m^2 < 0$, there is always a solution for which T grows exponentially at late times, and thus the late time behavior is modified by tachyon condensation. It is reasonable to require that in a sensible vacuum of string theory there should not be modes with $m^2 < 0$; this is certainly the case in the type II examples discussed in subsection 3.1.

For early times, all fields T grow with time (5.3). The question of (in)stability of the solution to generic perturbations takes one outside the realm of the linear dilaton part of spacetime, and we will not discuss it here.

5.2 Infrared divergences at one loop

In this subsection, we would like to show that modes (5.3) with $\vec{k}^2 + m^2 < Q^2$, which correspond to exponentially growing wavefunctions (5.6) at early and late times, give rise to infrared divergences in one loop amplitudes in string theory in asymptotically timelike linear dilaton spacetimes. In fact, we have seen this already: in the discussion of the bosonic string on such backgrounds we pointed out that these modes contribute to the partition sum the term (3.18), which can be written as $\exp(-\pi\tau_2(\vec{k}^2 + m^2 - Q^2))$. Thus, if $m^2 + \vec{k}^2 < Q^2$, the integral over τ_2 diverges from the region near $\tau_2 = \infty$, which is an infrared divergence, usually associated with a tachyon. A similar structure arises for the type II case (3.22). As in the case of a constant dilaton, one can try to define the divergent integral by analytic continuation [56] and interpret the resulting finite, complex result as giving a decay rate [56, 57].

In the rest of this subsection we will discuss the one loop amplitude in field theory. The two main motivations for this discussion are the following. First, in the next section we will study the case of de Sitter space, which does not seem to have a satisfactory string theory realization. There, we will have to use field theoretic techniques. Second, we would like to discuss the relation of the euclidean worldsheet partition sums (3.17), (3.22) to existing

[12] For simplicity, we include only the self-interactions of T; similar comments apply to interactions of T with other fields.

calculations in the QFT literature which are done in Minkowski space, and appear to be infrared finite.

As a warm-up exercise, consider the one-loop vacuum diagram of a scalar field in Minkowski space (with constant dilaton). In Schwinger parametrization, it reads

$$Z_{1-\text{loop}} = \text{Tr} \int_0^\infty \frac{ds}{s} e^{-is(-\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2 - i\epsilon)} , \tag{5.11}$$

where the trace runs over an orthonormal basis of delta-function normalizable modes, which we choose to be the following eigenfunctions of the wave operator $-\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2$:

$$\left\{ e^{iEt + i\vec{k}\cdot\vec{x}} | (E, \vec{k}) \in \mathbb{R}^{d+1} \right\} . \tag{5.12}$$

Thus we find

$$Z_{1-\text{loop}} = \int dE \int d^d k \int_0^\infty \frac{ds}{s} e^{-is(-E^2 + \vec{k}^2 + m^2 - i\epsilon)} . \tag{5.13}$$

Note that the $i\epsilon$ perscription makes the integral converge at large s. Consider first the case $m^2 \geq 0$. Then it is convenient to Wick rotate (see e.g. [9], p. 83)

$$E \mapsto iE ; \qquad s \mapsto -is , \tag{5.14}$$

so that (5.13) becomes

$$Z_{1-\text{loop}} = i \int dE \int d^d k \int_0^\infty \frac{ds}{s} e^{-s(E^2 + \vec{k}^2 + m^2)} . \tag{5.15}$$

This expression clearly has no infrared (large s) divergences, like the original Minkowski amplitude. Recall, also, that in string theory the parameter s in (5.15) is proportional to (the imaginary part of) the modulus of the worldsheet torus, τ_2.

For the case $m^2 < 0$, it is not a priori clear that performing the Wick rotation (5.14) is a sensible thing to do. In particular, while (5.13) does not have large s divergences, the Wick rotated expression (5.15) is IR divergent in this case. Thus, one might be tempted to perform the continuation (5.14) for all modes with positive m^2, and leave the contributions of tachyonic modes in the original form (5.13). One problem with this is that one would then have to treat differently regions in the spatial momentum integral where $\vec{k}^2 + m^2$ is positive and negative.

In string theory[13] one is instructed to perform the continuation to (5.15) for all modes, including those with $m^2 < 0$. All but at most a few of the string modes have positive m^2, and the treatment of the remaining ones is determined by the requirement of modular invariance.

Another way to see this is the following. Suppose in string theory one performed the continuation (5.14) for all the positive m^2 modes, and treated the negative m^2 modes differently. The resulting partition sum $Z(\tau)$ (see e.g. (3.22)) would not be modular invariant, and there would not be any justification to integrate it over the modular domain which excludes the small s region. One would thus integrate over s between zero and infinity. The

[13] E.g. the critical bosonic string, or type 0 string theory.

resulting amplitude would have a divergence from *small* s, which would be equivalent [41] to the original infrared divergence due to the tachyon. Thus, in string theory, one cannot avoid the infrared divergence associated with (5.15) with $m^2 < 0$. The best one can do is to push it into a UV regime, but the two are related by the standard UV/IR duality of perturbative string theory (worldsheet duality).

It is elementary to generalize the above discussion to the timelike linear dilaton case. In the presence of a dilaton $\Phi = -Qt$, the one loop partition sum[14] takes the form

$$Z_{1-\text{loop}} = \text{Tr} \int_0^\infty \frac{ds}{s} e^{-is(-\eta^{\mu\nu}\partial_\mu\partial_\nu + 2Q\partial_0 + m^2 - i\epsilon)}, \tag{5.16}$$

where the trace runs over a suitable basis of normalizable functions. We choose the basis

$$\left\{ e^{-Qt} e^{iEt + i\vec{k}\cdot\vec{x}} | (E, \vec{k}) \in \mathbb{R}^{d+1} \right\}, \tag{5.17}$$

such that (5.16) becomes

$$\begin{aligned} Z_{1-\text{loop}} &= \int dE \int d^d k \int_0^\infty \frac{ds}{s} e^{-is(-E^2 + \vec{k}^2 + m^2 - Q^2 - i\epsilon)} \\ &= i \int dE \int d^d k \int_0^\infty \frac{ds}{s} e^{-s(E^2 + \vec{k}^2 + m^2 - Q^2)} \\ &= i\pi^{(d+1)/2} \int_0^\infty \frac{ds}{s} s^{-(d+1)/2} e^{-s(m^2 - Q^2)}, \end{aligned} \tag{5.18}$$

where we again rotated the contour of integration over s and E as in (5.14). We see that this indeed has a large s divergence for all $m^2 < Q^2$. The treatment of these modes is again determined by string theory, as above. This has been used implicitly in arriving at (3.17), (3.22).

6. De Sitter space

We would now like to repeat the analysis of section 5 for de Sitter space (see e.g. [7], [58]–[64] and references therein for some recent discussions of de Sitter space). In global coordinates, which cover the whole de Sitter manifold, the metric reads

$$ds^2 = -d\tau^2 + l^2 \cosh^2\left(\frac{\tau}{l}\right) d\Omega_d^2, \tag{6.1}$$

where l is the de Sitter radius, $-\infty < \tau < \infty$, and $d\Omega_d^2$ is the line element on the unit d-sphere. This metric solves Einstein's equations with positive cosmological constant

$$\Lambda = \frac{d(d-1)}{2l^2} \tag{6.2}$$

and describes a space that contracts for early times τ and expands for late times. In that respect, it is similar to the Einstein frame metric describing the solutions of subsection 2.2 (see (2.11), (2.15) for the asymptotic behavior of this Einstein frame metric). We will focus

[14] Other one loop amplitudes will behave in the same way for the purposes of the present discussion.

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on the early time region $\tau \to -\infty$. It will be convenient to describe this region in planar coordinates, which cover half of de Sitter space:

$$ds^2 = -dt^2 + e^{-2Ht} d\vec{x}^2 \,. \tag{6.3}$$

Here $-\infty < t < \infty$, $H = l^{-1}$ is the 'Hubble constant', and $\vec{x}$ are coordinates on a d-plane. Early times correspond to $t \to -\infty$.

Translations of t have to be accompanied by dilations in order to preserve the metric. The associated Killing vector K is

$$K = \partial_t + H x^i \partial_i \,, \tag{6.4}$$

which has norm squared

$$(K, K) = -1 + H^2 e^{-2tH} \vec{x}^2 \,. \tag{6.5}$$

Thus we see that K is timelike whenever $H^2 e^{-2tH} \vec{x}^2 < 1$, so that there is a cosmological event horizon of size

$$\vec{x}^2 = H^{-2} e^{2tH} \,. \tag{6.6}$$

In analogy with subsection 5.1, we now study the classical evolution of a scalar field with mass m in the background (6.3):

$$S = \frac{1}{2} \int dt \int d^d x \, e^{-dHt} \left(\dot{T}^2 - e^{2Ht} (\nabla T)^2 - m^2 T^2 \right) , \tag{6.7}$$

with equation of motion

$$\ddot{T} - dH\dot{T} - e^{2Ht} \nabla^2 T + m^2 T = 0 \,. \tag{6.8}$$

We want to study this scalar field for early times t, for which the $e^{2Ht} \nabla^2 T$ term in (6.8) is negligible.[15] Then the solutions are

$$T_\pm \sim e^{h_\pm t} e^{i\vec{k}\cdot\vec{x}} \,, \qquad t \to -\infty \,, \tag{6.9}$$

where

$$h_\pm = \frac{dH}{2} \pm \sqrt{\left(\frac{dH}{2}\right)^2 - m^2} \,. \tag{6.10}$$

For $0 < m^2 < (\frac{dH}{2})^2$, both solutions grow exponentially with t for early times. One can again rescale the field T in (6.7), $T = \Psi \exp(dHt/2)$, and find that the action (6.7) (with the second term and a total derivative neglected) takes the form

$$S = \frac{1}{2} \int dt \int d^d x \left(\dot{\Psi}^2 - m_{\text{eff}}^2 \Psi^2 \right) , \tag{6.11}$$

with $m_{\text{eff}}^2 = m^2 - (dH/2)^2$. Hence, growing modes exist when

$$m^2 < \left(\frac{dH}{2}\right)^2 . \tag{6.12}$$

[15] We will comment on what this means shortly.

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In the analysis leading to (6.9), we assumed that t was sufficiently large and negative for the $e^{2Ht}\nabla^2 T$ term in (6.8) to be negligible compared to the other two terms in the action. This is the case if

$$e^{2Ht}\vec{k}^2 \ll m^2 \,, \qquad \left(\frac{dH}{2}\right)^2 . \tag{6.13}$$

Using (6.6), we see that our approximation is valid only for modes with wavelength large compared to the horizon radius (up to a d-dependent factor). This should be contrasted with discussions like that of [65], where fluctuations inside the event horizon are discussed.

As in subsection 5.2, the growing modes (6.12) lead to infrared divergences in one-loop amplitudes. The contribution of the scalar field (6.11) to the one-loop vacuum amplitude is

$$Z_{1-\text{loop}} = \text{Tr} \int_0^\infty \frac{ds}{s} e^{-is(-\partial_t^2 + dH\partial_t + e^{2Ht}\nabla^2 + m^2 - i\epsilon)} \,, \tag{6.14}$$

where the trace runs over a suitable basis of normalizable functions. We choose these functions to be eigenfunctions of the differential operator in the exponent in (6.14), and label them by their behavior for early t, which is given by

$$\left\{ e^{\frac{dHt}{2}} e^{iEt + i\vec{k}\cdot\vec{x}} | (E, \vec{k}) \in \mathbb{R}^{d+1} \right\} . \tag{6.15}$$

We evaluate the trace in the early time region, so that the $e^{2Ht}\nabla^2$ term in (6.14) is negligible. Then (6.14) becomes

$$Z_{1-\text{loop}} = \int dE \int d^d k \int_0^\infty \frac{ds}{s} e^{-is(-E^2 + m^2 - (\frac{dH}{2})^2 - i\epsilon)} \,, \tag{6.16}$$

where the integral over $\vec{k}$ has a large $|\vec{k}|$ cutoff determined by the condition that the $e^{2Ht}\nabla^2$ term in (6.14) should be negligible. The result of the integral over $\vec{k}$ is a factor converting the coordinate volume of space (which multiplies the right hand side of (6.16), although we have suppressed it in the formulae) into the physical volume of space (i.e., the volume measured with the metric (6.3)). Omitting this volume factor, we obtain by Wick rotation (as in subsection 5.2)

$$Z_{1-\text{loop}} = i \int dE \int_0^\infty \frac{ds}{s} e^{-s(E^2 + m^2 - (\frac{dH}{2})^2)} \,. \tag{6.17}$$

We see that (6.17) has large s divergences for all masses with $m^2 < (\frac{dH}{2})^2$. This is precisely the range of masses corresponding to growing modes (6.12).

7. Summary and discussion

The main purpose of this paper was to study string propagation in the time-dependent spacetime (2.31), (2.32), which includes a cosmological singularity, in the vicinity of which spacetime has the form $\mathbb{R}^{1,1}/\mathbb{Z}$. In such spacetimes there are many qualitative and quantitative issues that are not well understood in the framework of QFT in curved spacetime. For example, it is not clear whether one should only include the part of the spacetime which looks like a circle expanding from zero size at the singularity to a finite size at late times, or also the pre-big bang region, as well as the other regions in figures 1, 2. There is a question as to the nature of observables, and in particular the continuation of wavefunctions through cosmological singularities. Ultimately, one would like to compute S-matrix elements in such spacetimes, and study the effects of the cosmological singularities on them.

We found that many of these and other questions can be addressed by realizing the cosmological spacetime (2.31), (2.32) as a coset CFT of the form $SL(2,\mathbb{R})/(U(1)\times\mathbb{Z})$. Our main results are the following:

The coset CFT describes a spacetime consisting of six regions (see figures 1, 2), four of which include asymptotic early and late time regimes (two of each kind). The observables correspond to scattering states prepared at early times, and their S-matrix elements to evolve to some particular states at late times.

The asymptotic past consists of two disconnected regions, regions I and VI in figures 1, 2. Correspondingly, the Hilbert space of asymptotic past states naturally takes the form of a direct product of the two Hilbert spaces corresponding to the two regions. Similarly, the future Hilbert space is a direct product of the Hilbert spaces corresponding to regions II and V. In the asymptotic past and future the solution approaches a timelike linear dilaton one. The string coupling goes to zero, which makes it relatively easy to construct the incoming and outgoing Hilbert spaces.

Embedding the cosmological spacetime in $SL(2,\mathbb{R})$ allows one to continue wavefunctions through singularities. We used this continuation to compute the Bogolubov coefficients relating particles in the initial and final Hilbert spaces. We found that in order to obtain unitary evolution in the cosmological spacetime, one has to include both of the incoming and both of the outgoing regions. We also showed that the incoming vacuum evolves to a state with particles in the far future.

We discussed the existence of growing modes in asymptotically timelike linear dilaton solutions. We showed that scalar fields with mass $m^2 < Q^2$ have exponentially growing or decaying wavefunctions (for low enough momentum), and lead to infrared divergences in one loop amplitudes.

We briefly discussed the relation of our results to gravity in de Sitter space. In the early and late time regimes, the qualitative structure of solutions to the wave equation in timelike linear dilaton and global de Sitter spacetime is similar, and thus one can use some of the arguments made in the timelike linear dilaton case. One finds that fields with mass smaller than the Hubble mass lead to modes with growing wavefunction in de Sitter space, which are analogous to those with $m^2 < Q^2$ in timelike linear dilaton backgrounds. In particular, they lead to infrared divergences in one loop diagrams.

An issue that has received some attention recently is the question of holography in de Sitter space, and in particular the proposal [3] that gravity in de Sitter space is equivalent to a euclidean CFT living on the spacelike boundary at early and late times. The analogous statement in asymptotically timelike linear dilaton spacetimes would be that the theory is equivalent to a euclidean theory living on the spacelike boundary at $|t| \to \infty$.

From our discussion it seems that the situation in both of these cases is expected to be more analogous to gravity in asymptotically flat spacetime than to that in anti-de Sitter or spacelike linear dilaton vacua. In the latter case, there is a class of observables that correspond to non-normalizable wavefunctions supported at the boundary, and one can study the path integral of gravity as a function of these "boundary conditions". In flat spacetime, one instead studies (delta function) normalizable wavefunctions that correspond to scattering states and their S-matrix.

In de Sitter and timelike linear dilaton spacetimes, the analogues of the non-normalizable wavefunctions of AdS and spacelike linear dilaton solutions seem to be the growing modes discussed in sections 5, 6. The "good observables" are in fact analogues of the scattering states in flat spacetime. Therefore, it is not completely clear that a picture in terms of a euclidean theory living on the spacelike boundary will be more useful in this case than an analogous picture in flat spacetime. A better understanding of holography in these spacetimes would be interesting.

Many other issues deserve further study. We described in section 2 a large class of solutions, the generalized Kasner solutions (2.23), (2.24). If the parameters a_i (2.22) are small, one can think of the resulting solutions as "small perturbations" of the generalized Milne solution that we discussed in sections 3, 4 using coset CFT techniques. These perturbations correspond to zero momentum modes of the graviton and dilaton, which are massless but become effectively tachyonic in the timelike linear dilaton spacetime, (3.13). The Kasner solutions are in fact an example of how such modes can significantly influence the evolution, even if they are very small at early times. The uncompactified generalized Milne solution (2.26) is non-singular at $t = 0$ (as discussed in subsection 2.3), while arbitrarily small perturbations of it at early times, corresponding to turning on $a_2, a_3, \dots$ lead to a singularity at finite t. It would be interesting to understand whether all the generalized Kasner solutions can be understood by thinking of the spacetime as a perturbed coset model. In particular, it would be interesting to understand whether one still has to include the different regions that were seen to play a role in the generalized Milne case, and if so, how to continue wavefunctions through the Kasner singularities.

Another general question involves the supercritical type II solutions corresponding to asymptotically timelike linear dilaton spacetimes at $t \to +\infty$, discussed in [20, 39] and in subsection 3.1. Since these solutions do not contain fields with $m^2 < 0$, it seems that their late time behavior might be stable. One can then ask what the status of these solutions is within M-theory. They are obviously higher than eleven dimensional, and at least naively seem to have more degrees of freedom than other known vacua of M-theory. It is not clear how they are related to the standard eleven dimensional descriptions of M-theory.

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